COT 6405 Introduction to Theory of Algorithms

Topic 8. Quicksort

Quicksort

- Sorts “in place”
  - Only a constant number of elements stored outside the sorted array
- Sorts O(n lg n) in the average case
- Sorts $O(n^2)$ in the worst case
- So why people use it instead of merge sort?
  - Merge sort does not sort “in place”

Quicksort: divide and conquer

- **Divide**: Array A[p..r] is partitioned into two non-empty subarrays A[p..q] and A[q+1..r]
  - All elements in A[p..q] are less than all elements in A[q+1..r]
- **Conquer:** The subarrays are recursively sorted by calls to quicksort
- **Combine:** No work is needed to combine the subarrays, because they are sorted in place.

Quicksort Code

```
Quicksort(A, p, r)
{
    if (p < r)
    {
        q = Partition(A, p, r);
        Quicksort(A, p, q-1);
        Quicksort(A, q+1, r);
    }
}  // what is the initial call?
```

Partition

- Clearly, all the actions take place in the **`partition()`** function
  - Rearranges the subarray “in place”
  - End result:
    - Two subarrays
    - All values in 1st subarray < all values in 2nd
  - Returns the index of the “pivot” element separating the two subarrays
- How do we implement this function?

Partitioning

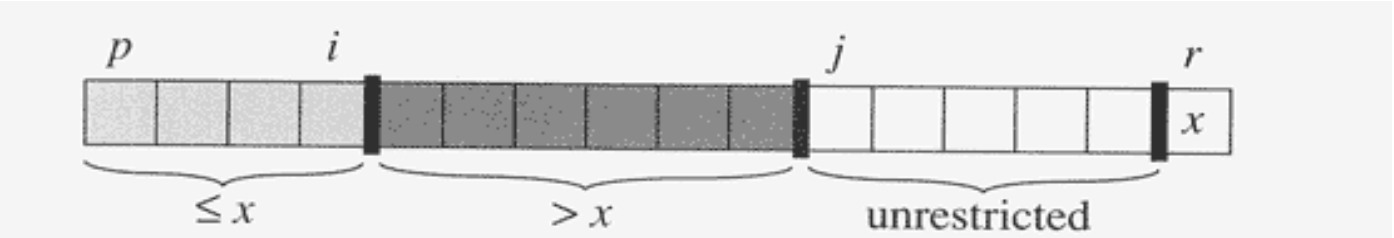
- PARTITION first selects the **pivot** (How?)
  - the last element $A[r]$ in the subarray $A[p \,..\, r]$
- The array is partitioned into four regions
  - some of which may be empty
- **Loop invariant:**

  1. All entries in $A[p \,..\, i]$ are ≤ pivot.

  2. All entries in $A[i + 1 \,..\, j - 1]$ are > pivot.

  3. $A[r]$ = pivot.

  4. It's not needed as part of the loop invariant, but the fourth region is $A[\, j \,..\, r-1]$, whose entries have not yet been examined.



Partition array $A[p..r]$

```
PARTITION(A, p, r)
        x ← A[r]          // select the pivot
        i ← p − 1
        for j ← p to r − 1
                if A[ j ] ≤ x
                        i ← i + 1
                        exchange A[i ] ↔ A[ j ]
        // move the pivot between the two subarraies
        exchange A[i + 1] ↔ A[r]
        // return the pivot
        return i + 1
```

What is the running time of `partition()`?

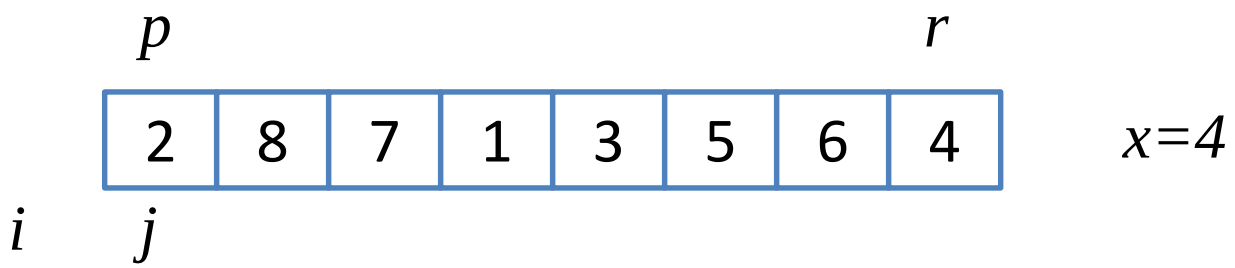
Correctness

- **Initialization:** Before the loop starts, all the conditions of the loop invariant are satisfied, because r is the pivot and the subarrays $A[p \,..\, i]$ and $A[i+1 \,..\, j-1]$ are empty.
- **Maintenance:** While the loop is running, if $A[j] \le$ pivot, then $A[j]$ and $A[i+1]$ are swapped and then i and j are incremented. If $A[j] >$ pivot, then increment only j .
- **Termination:** When the loop terminates, $j = r$, so all elements in A are partitioned into one of the 3 cases: $A[p \,..\, i] \le$ pivot, $A[i+1 \,..\, r-1] >$ pivot, and $A[r]$ = pivot.

The last two lines of PARTITION move the pivot element from the end of the array to between the two subarrays. This is done by swapping the pivot and the first element of the second subarray, i.e., by swapping $A[i+1]$ and $A[r]$

Time for partitioning: $\Theta(n)$ to partition an n-element subarray.

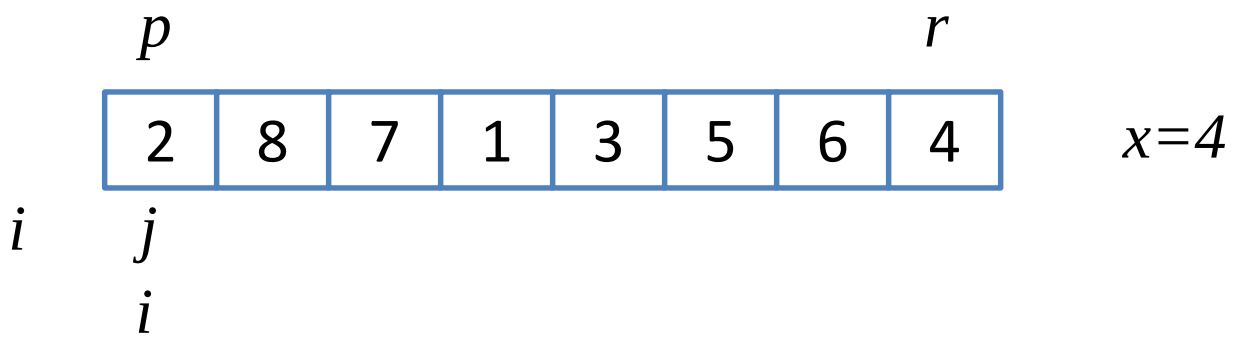
An example of Partition



PARTITION(*A*, *p*, *r*)

$x \leftarrow A[r]$ // select the pivot
$i \leftarrow p - 1$
for $j \leftarrow p$ to $r - 1$
if $A[j] \leq x$
$i \leftarrow i + 1$
exchange $A[i] \leftrightarrow A[j]$
exchange $A[i + 1] \leftrightarrow A[r]$
return $i + 1$

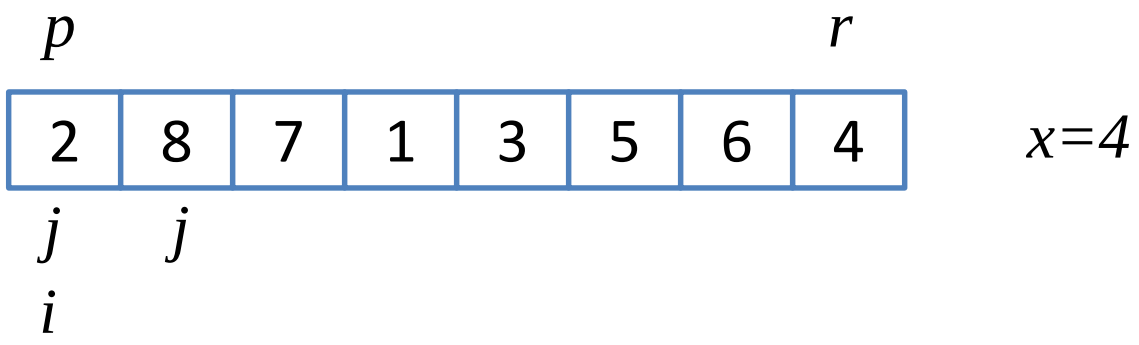
An example of Partition (cont'd)



PARTITION(*A*, *p*, *r*)

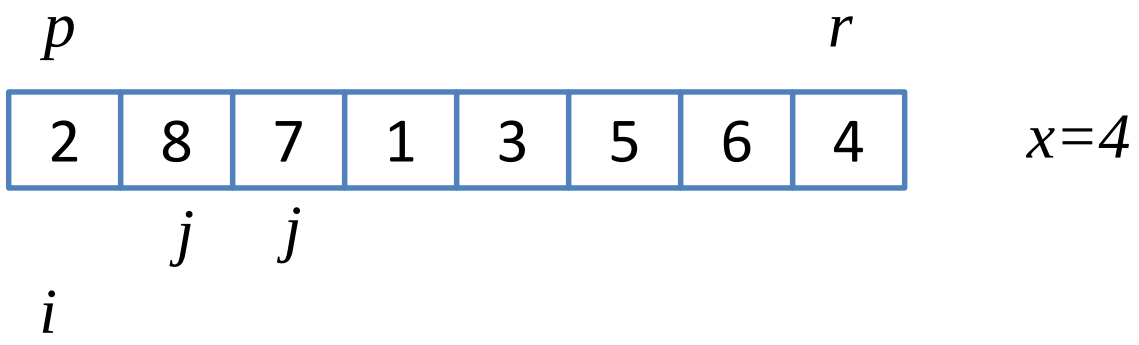
```
x ← A[r]          // select the pivot
i ← p − 1
for j ← p to r − 1
        if A[ j ] ≤ x
                i ← i + 1
                exchange A[i ] ↔ A[ j ]
exchange A[i + 1] ↔ A[r]
return i + 1
```

An example of Partition (cont'd)




```
PARTITION(A, p, r)
        x ← A[r]          // select the pivot
        i ← p − 1
        for j ← p to r − 1
                if A[ j ] ≤ x
                        i ← i + 1
                        exchange A[i ] ↔ A[ j ]
        exchange A[i + 1] ↔ A[r]
        return i + 1
```

An example of Partition (cont'd)

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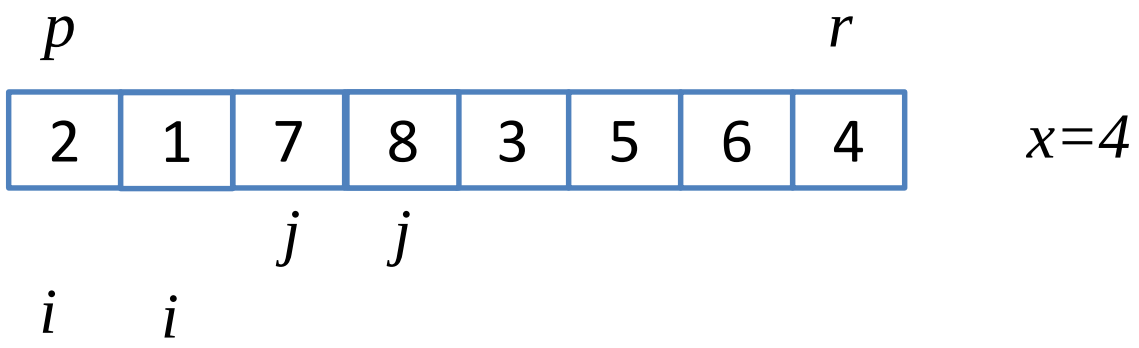
        x ← A[r]          // select the pivot
        i ← p − 1
        for j ← p to r − 1
                if A[ j ] ≤ x
                        i ← i + 1
                        exchange A[i ] ↔ A[ j ]
        exchange A[i + 1] ↔ A[r]
        return i + 1
```

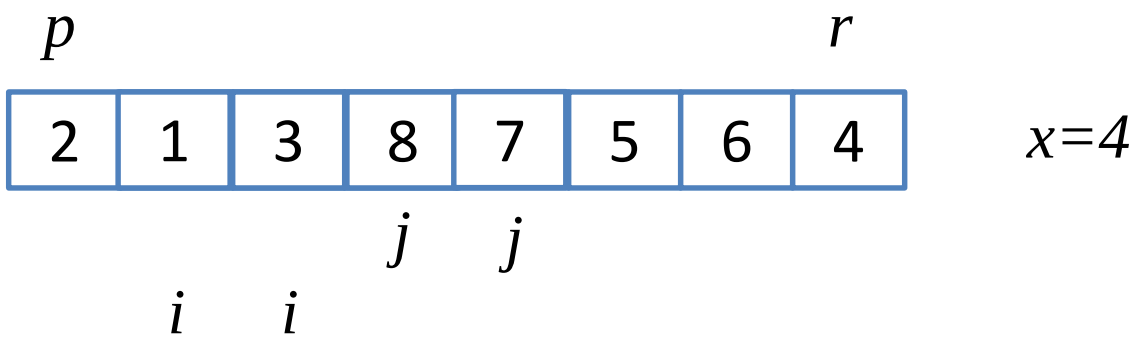
An example of Partition (cont'd)



PARTITION(*A*, *p*, *r*)

```
x ← A[r]          // select the pivot
i ← p − 1
for j ← p to r − 1
        if A[ j ] ≤ x
                i ← i + 1
                exchange A[i ] ↔ A[ j ]
exchange A[i + 1] ↔ A[r]
return i + 1
```

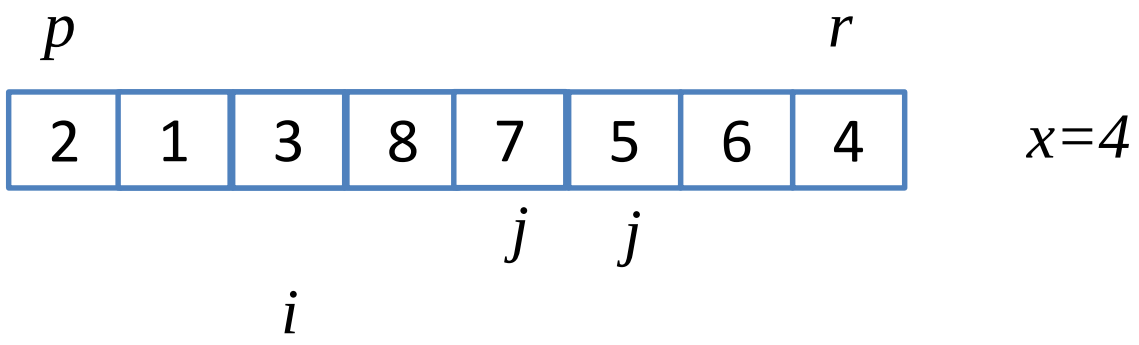
An example of Partition (cont'd)



PARTITION(*A*, *p*, *r*)

```
x ← A[r]          // select the pivot
i ← p − 1
for j ← p to r − 1
        if A[ j ] ≤ x
                i ← i + 1
                exchange A[i ] ↔ A[ j ]
exchange A[i + 1] ↔ A[r]
return i + 1
```

An example of Partition (cont'd)




```
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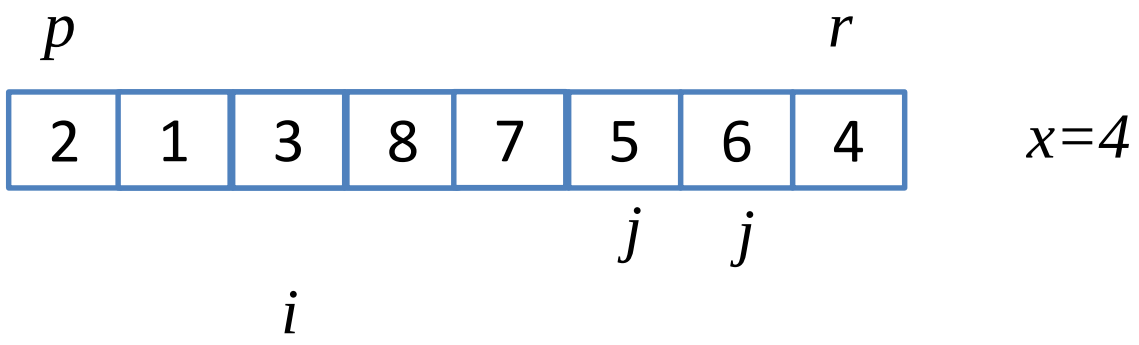
        i ← p − 1
        for j ← p to r − 1
                if A[ j ] ≤ x
                        i ← i + 1
                        exchange A[i ] ↔ A[ j ]
        exchange A[i + 1] ↔ A[r]
        return i + 1
```

An example of Partition (cont'd)

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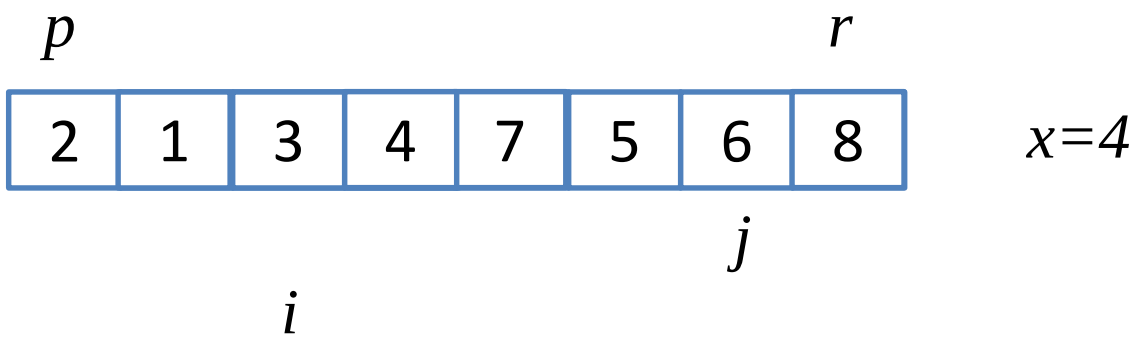
        i ← p − 1
        for j ← p to r − 1
                if A[ j ] ≤ x
                        i ← i + 1
                        exchange A[i ] ↔ A[ j ]
        exchange A[i + 1] ↔ A[r]
        return i + 1
```

An example of Partition (cont'd)

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        i ← p − 1
        for j ← p to r − 1
                if A[ j ] ≤ x
                        i ← i + 1
                        exchange A[i ] ↔ A[ j ]
        exchange A[i + 1] ↔ A[r]
        return i + 1
```

Analyzing Quicksort

- Worst-case performance
  - The worst-case behavior for quicksort occurs when the partitioning routine produces with n-1 elements and one with 0 elements
- The recurrence is
  - $T(n) = T(n-1) + T(0) + \Theta(n)$

    $= T(n-1) + \Theta(n)$

Exercise

- For T(n) = T(n-1) + $\Theta(n)$, use substitution method to show that the $\mathrm{T(n)} = O(n^2)$.

Exercise (cont'd)

- T(n) = T(n-1) + $\Theta(n)$
- Basis: n = 1, T(1) = Θ (1)
  Inductive step: suppose T(k) $\leq$ ck^2 for all k < n, then

  T(n) $\leq$ c$(n-1)^2$ + $c'n$

  $=$ cn^2- 2cn + c +$c'n$

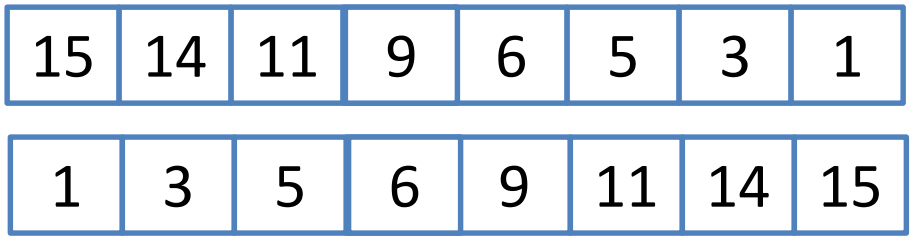
  = cn^2 - (2c- c')n + c

  $\leq$ cn^2 if (2c-c')n + c ≥ 0 - > n_0= 1 and c' $\leq$ 2c

  Thus, T(n) = O(n^2)

A question

- Will any particular input elicit the worst case?
  - Yes, the array is already sorted in the reverse order
  - Or it is already sorted




```
PARTITION(A, p, r)
        x ← A[r]          // select the pivot
        i ← p − 1
        for j ← p to r − 1
                if A[ j ] ≤ x
                        i ← i + 1
                        exchange A[i ] ↔ A[ j ]
        exchange A[i + 1] ↔ A[r]
        return i + 1
```

```
Quicksort(A, p, r)
{
    if (p < r)
    {
        q = Partition(A, p, r);
        Quicksort(A, p, q-1);
        Quicksort(A, q+1, r);
    }
}
```

Best-case performance

- The best-case behavior occurs when Partition() produces two sub-problems, each of size no more that n/2.
- The recurrence for the running time is
  - T(n) = 2T(n-1/2) + $\Theta(n)$
  - By case 2 of the master theorem, T(n) = Θ(nlgn)

Performance of quicksort

- The running time of quicksort depends on the partitioning of the subarrays:
  - If the subarrays are balanced, then quicksort can run as fast as mergesort.
  - If they are unbalanced, then quicksort can run as slowly as insertion sort.

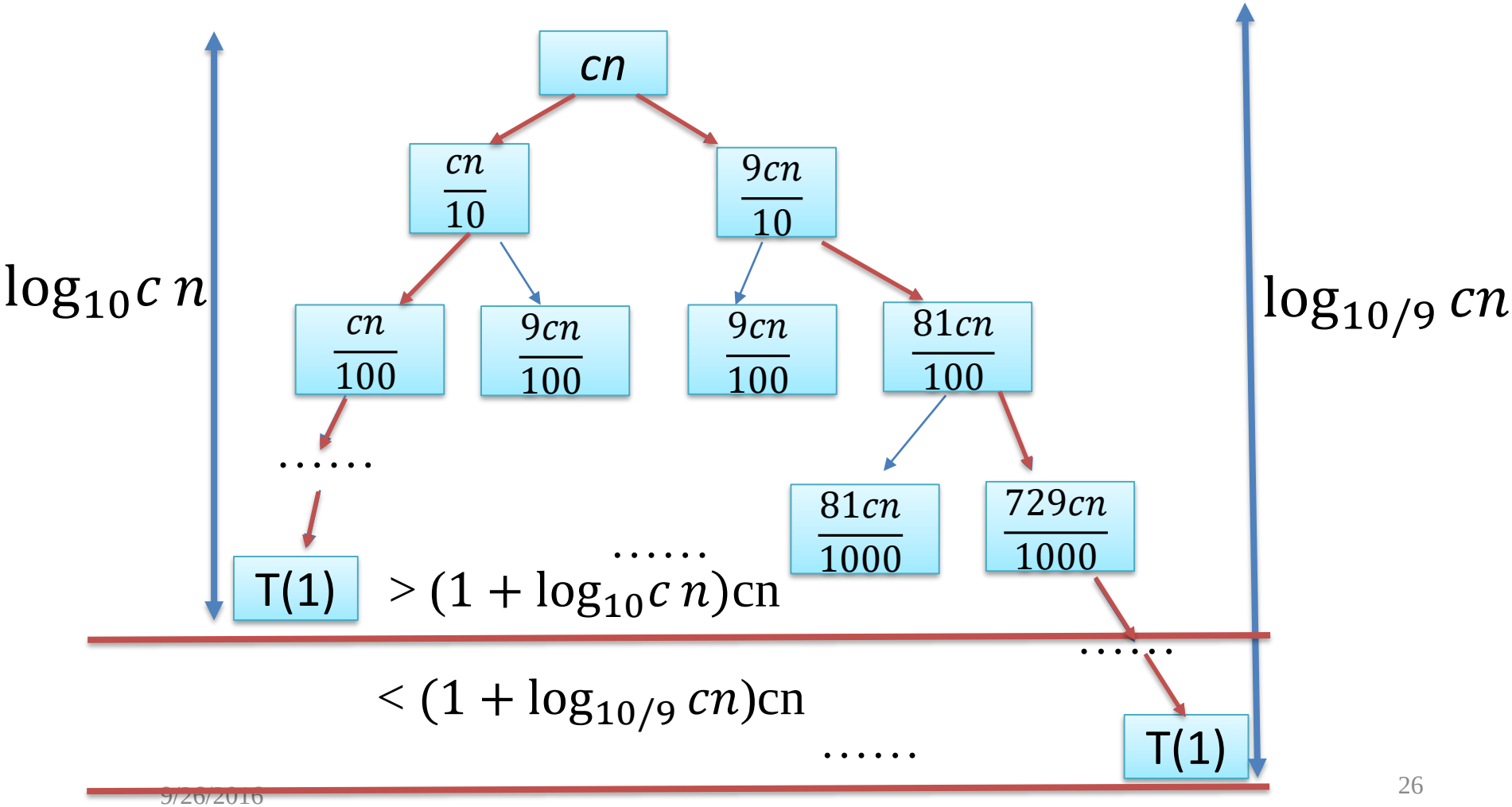
Analyzing Quicksort: Average Case

- Assuming random input → average-case running time is much closer to O(n lg n) than $O(n^2)$
- First, a more intuitive explanation/example:
  - Suppose that partition() always produces a 9-to-1 split. This looks quite unbalanced!
  - The recurrence is thus:

    $T(n) = T(9n/10) + T(n/10) + cn$
  - How deep will the recursion go? (draw it)

Average Case (cont'd)

- T(n) = T(9n/10) + T(n/10) + cn



Average Case (cont'd)

- For shortest path for the root to the leaf
  - The subproblem size for a node at depth *i* is $(\frac{1}{10})^i$cn
  - The subproblem size hits T(1), when $(\frac{1}{10})^i$cn= 1, or $i = \log_{10} cn$
  - Thus, the length of the shortest path is $\log_{10} cn$

Average Case (cont'd)

- For longest path for the root to the leaf
  - The subproblem size for a node at depth *i* is $(\frac{9}{10})^i$cn
  - The subproblem size hits T(1), when$(\frac{9}{10})^i$cn= 1, or $i = \log_{10/9} cn$
  - Thus, the length of the longest path is $\log_{10/9} cn$

Average Case (cont'd)

- Notice that every level of the tree has a cost of cn , until the recursion reaches a boundary condition at depth $\log_{10} cn = \Theta(lgn)$
- Then, the levels have cost at most cn
- The recursion terminates at depth $\log_{10/9} n = \Theta(lgn)$

Average Case (cont'd)

- The total cost of quicksort T(n)

  T(n) $> (1 + \log_{10} cn)$cn -> Ω (nlgn)

  T(n) $< (1 + \log_{10/9} cn)$cn -> O(nlgn)

  T(n) = $\Theta(\text{nlgn})$

Analyzing Quicksort: Average Case

- Intuitively, a real-life run of quicksort will produce a mix of “bad” and “good” splits
  - Randomly distributed among the recursion tree
  - Pretend for intuition that they alternate between best-case (n/2 : n/2) and worst-case (n-1 : 0)
  - What happens if we bad-split root node, then good-split the resulting size (n-1) node?
    - We end up with 3 subarrays, size 0, (n-1)/2-1, (n-1)/2
    - Combined cost of splits = n + n -1 = 2n -1 = $\Theta(n)$
    - No worse than if we had good-split the root node!
    - T(n) = 2T(n-1/2) + $\Theta(n)$ v.s. T(n) = T(0) + 2T(n-1/2) + $\Theta(n)$

Partition cost in Elliptical Shading

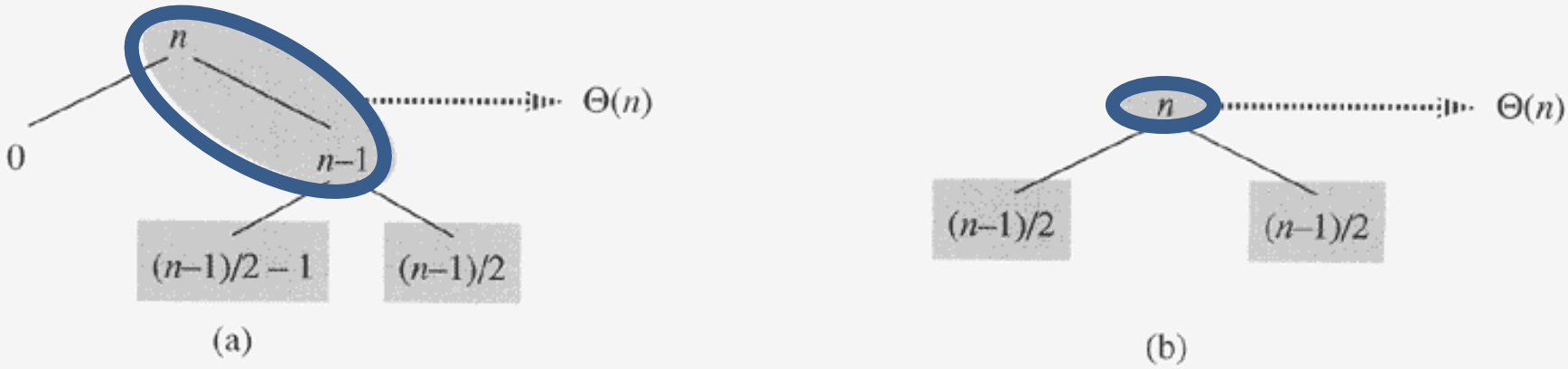


Figure 7.5 **(a)** Two levels of a recursion tree for quicksort. The partitioning at the root costs n and produces a "bad" split: two subarrays of sizes 0 and $n - 1$. The partitioning of the subarray of size $n - 1$ costs $n - 1$ and produces a "good" split: subarrays of size $(n - 1)/2 - 1$ and $(n - 1)/2$. **(b)** A single level of a recursion tree that is very well balanced. In both parts, the partitioning cost for the subproblems shown with elliptical shading is $\Theta(n)$. Yet the subproblems remaining to be solved in (a), shown with square shading, are no larger than the corresponding subproblems remaining to be solved in (b).

Analyzing Quicksort: Average case

- Intuitively, the O(n) cost of a bad split (or 2 or 3 bad splits) can be absorbed into the O(n) cost of each good split
- Thus running time of alternating bad and good splits is still O(n lg n), with slightly higher constants
- How can we be more rigorous?

Analyzing Quicksort: Average case

- For simplicity, assume:
  - All inputs distinct (no repeats)
- Partition around a random element
    - all splits (0:n-1, 1:n-2, 2:n-3, … , n-1:0) are equally likely
    - In general, a split can be represented by (k : n-1-k)
- What is the probability of a particular split happening?
- Answer: 1/n

Analyzing Quicksort: Average case

- So partition generates splits
  (0:n-1, 1:n-2, 2:n-3, … , n-2:1, n-1:0)
  each with probability 1/n
- T(n) is the expected running time, T(n) = ?

$$T(n) = \frac{1}{n}\sum_{k=0}^{k=n-1} T(k) + T(n-1-k) + \Theta(n)$$

- What is each term under the summation for?
- What is the Θ(n) term for?

Average case

- T(n) = $\frac{1}{n}\sum_{k=0}^{k=n-1} T(k) + T(n-1-k) + \Theta(n)$
- We can rewrite the above equation as
- T(n) = $\frac{2}{n}\sum_{k=0}^{k=n-1} T(k) + \Theta(n)$

Why?

- T(n) =$\frac{1}{n}$(T(0)+T(n-1)+T(1)+T(n-2)+…+ T(n-1)+T(0))

Average case (cont'd)

- We can solve this recurrence using the dreaded substitution method
  - Guess the answer
    - T(n) = O(n lg n)
  - Assume that the inductive hypothesis holds
    - What's the inductive hypothesis?
    - T(*k*) $\leq$ *a k* lg *k* + b for some constants *a>0* and *b>0 and k<n*

Average case (cont'd)

- The recurrence to be solved
  - T(n) = $\frac{2}{n}\sum_{k=0}^{k=n-1} T(k) + \Theta(n)$
- What next?
  - Plug in the inductive hypothesis
  - $T(n) \leq \frac{2}{n}\sum_{k=0}^{k=n-1}(aklgk + b) + \Theta(n)$

Average case (cont'd)

- The recurrence to be solved
  - $T(n) \leq \frac{2}{n}\sum_{k=0}^{k=n-1}(aklgk + b) + \Theta(n)$
- What next?
  - Expand out the k=0 case
  - For simplicity, when *n* = 0, we define
    - $anlgn = \lim_{n\to 0} anlgn = 0$
  - T(n) $\leq \frac{2}{n}\left[b + \sum_{k=1}^{k=n-1}(aklgk + b)\right] + \Theta(n)$

Average case (cont'd)

- The recurrence to be solved
  - $T(n) \leq \frac{2}{n}\left[b + \sum_{k=1}^{k=n-1}(aklgk + b)\right] + \Theta(n)$

    $= \frac{2}{n}\left[\sum_{k=1}^{k=n-1}(aklgk + b)\right] + \frac{2b}{n} + \Theta(n)$
- 2b/n is just a constant, so fold it into Θ(n)
  - $T(n) \leq \frac{2}{n}\sum_{k=1}^{k=n-1}(aklgk + b) + \Theta(n)$

Average case (cont'd)

- The recurrence to be solved
  - $T(n) \leq \frac{2}{n}\sum_{k=1}^{k=n-1}(aklgk + b) + \Theta(n)$
- What next?
  - Distribute the summation
  - $T(n) = \frac{2}{n}\sum_{k=1}^{k=n-1} aklgk + \frac{2}{n}\sum_{k=1}^{k=n-1} b + \Theta(n)$

Average case (cont'd)

- The recurrence to be solved
  - $T(n) = \frac{2}{n}\sum_{k=1}^{k=n-1} aklgk + \frac{2}{n}\sum_{k=1}^{k=n-1} b + \Theta(n)$
- What next?
  - Evaluate the summation
  - $T(n) = \frac{2}{n}\sum_{k=1}^{k=n-1} aklgk + \frac{2}{n}\sum_{k=1}^{k=n-1} b + \Theta(n)$

$$= \frac{2a}{n}\sum_{k=1}^{k=n-1} klgk + \frac{2b}{n}(n-1) + \Theta(n)$$

Average case (cont'd)

- The recurrence to be solved
  - T(n) = $\frac{2a}{n}\sum_{k=1}^{k=n-1} klgk + \frac{2b}{n}(n-1) + \Theta(n)$
- What next?
  - Since n-1<n, 2b(n-1)/n < 2b
  - T(n) $\leq \frac{2a}{n}\sum_{k=1}^{k=n-1} klgk + 2b + \Theta(n)$

Average case (cont'd)

$$T(n) \leq \frac{2a}{n} \sum_{k=1}^{k=n-1} klgk + 2b + \Theta(n)$$

It can be proved that

$$\sum_{k=1}^{k=n-1} klgk \leq \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$$

This summation gets its own set of slides later

Average case (cont'd)

- The recurrence to be solved
  - T(n) $\leq \frac{2a}{n}\sum_{k=1}^{k=n-1} klgk + 2b + \Theta(n)$
- What next?
  - Substitute $\sum_{k=1}^{k=n-1} klgk \leq \frac{1}{2}n^2 \lg n - \frac{1}{8}n^2$
  - T(n) $\leq \frac{2a}{n}\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right) + 2b + \Theta(n)$

Average case (cont'd)

- The recurrence to be solved
  - $T(n) \leq \frac{2a}{n}\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right) + 2b + \Theta(n)$
- What next?
  - Distribute the (2a/n) term
  - $T(n) \leq anlgn - \frac{an}{4} + 2b + \Theta(n)$

Average case (cont'd)

- The recurrence to be solved
  - $T(n) \leq anlgn - \frac{an}{4} + 2b + \Theta(n)$
- What is our goal?
  - Our goal is to get $T(n) \leq anlg\ n + b$
  - We rewrite T(n) as

$$T(n) \leq anlgn + b + [\Theta(n) + b - \frac{an}{4}]$$

Average case (cont'd)

- $T(n) \leq anlgn + b + [\Theta(n)+b - \frac{an}{4}]$
- What next?
  - Pick a large enough that $an/4$ dominates $\Theta(n)+b$
  - Then, $T(n) \leq anlgn + b$

Average case summary

$$T(n) = \frac{2}{n}\sum_{k=0}^{k=n-1} T(k) + \Theta(n)$$

$$\leq \frac{2}{n}\sum_{k=0}^{k=n-1}(aklgk + b) + \Theta(n)$$

$$= \frac{2}{n}\left[b + \sum_{k=1}^{k=n-1}(aklgk + b)\right] + \Theta(n)$$

$$= \frac{2}{n}\left[\sum_{k=1}^{k=n-1}(aklgk + b)\right] + \frac{2b}{n} + \Theta(n)$$

$$= \frac{2}{n}\sum_{k=1}^{k=n-1}(aklgk + b) + \Theta(n)$$

(when n -> ∞, $\frac{2b}{n}$ - > 0)

Average case summary(cont'd)

$$= \frac{2}{n}\sum_{k=1}^{k=n-1}(aklgk + b) + \Theta(n)$$

$$= \frac{2}{n}\sum_{k=1}^{k=n-1} aklgk + \frac{2}{n}\sum_{k=1}^{k=n-1} b + \Theta(n)$$

$$= \frac{2a}{n}\sum_{k=1}^{k=n-1} klgk + \frac{2b}{n}(n-1) + \Theta(n)$$

$$\leq \frac{2a}{n}\sum_{k=1}^{k=n-1} klgk + 2b + \Theta(n)$$

Average case summary (cont'd)

$$T(n) \leq \frac{2a}{n}\sum_{k=1}^{k=n-1} klgk + 2b + \Theta(n)$$

$$\leq \frac{2a}{n}\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right) + 2b + \Theta(n)$$

$$= an\lg n - \frac{an}{4} + 2b + \Theta(n)$$

$$= an\lg n + b + \Theta(n) + b - \frac{an}{4}$$

$$\leq an\lg n + b$$

Pick *a* large enough that *an*/4 dominates Θ(n)+b

Average case (cont'd)

- So T(n) $\leq$ an lg n + b for certain a and b
  - Thus the induction holds
  - Thus T(n) = O(n lg n)
  - Thus quicksort runs in O(n lg n) time on average
- Now let's prove the summation

$$\sum_{k=1}^{k=n-1} klgk \leq \frac{1}{2}n^2\lg n - \frac{1}{8}n^2$$

Tightly Bounding

- Prove $\sum_{k=1}^{k=n-1} klgk \leq \frac{1}{2}n^2 \lg n - \frac{1}{8}n^2$
- Split the summation for a tighter bound

$$\sum_{k=1}^{k=n-1} klgk = \sum_{k=1}^{k=\lceil \frac{n}{2} \rceil - 1} klgk + \sum_{k=\lfloor \frac{n}{2} \rfloor}^{k=n-1} klgk$$

Tightly Bounding (cont'd)

- $\sum_{k=1}^{k=n-1} klgk = \sum_{k=1}^{k=\lceil\frac{n}{2}\rceil-1} klgk + \sum_{k=\lceil\frac{n}{2}\rceil}^{k=n-1} klgk$
- The lgk in the second term is bounded by lg n

$$\sum_{k=1}^{k=n-1} klgk \leq \sum_{k=1}^{k=\lceil\frac{n}{2}\rceil-1} klgk + \sum_{k=\lceil\frac{n}{2}\rceil}^{k=n-1} klgn$$

Tightly Bounding (cont'd)

- $\sum_{k=1}^{k=n-1} klgk \leq \sum_{k=1}^{k=\lceil \frac{n}{2} \rceil -1} klgk + \sum_{k=\lceil \frac{n}{2} \rceil}^{k=n-1} klgn$
- Move the lgn outside the summation

$$\sum_{k=1}^{k=n-1} klgk \leq \sum_{k=1}^{k=\lceil \frac{n}{2} \rceil -1} klgk + \text{lgn} \sum_{k=\lceil \frac{n}{2} \rceil}^{k=n-1} k$$

Tightly Bounding (cont'd)

- $\sum_{k=1}^{k=n-1} klgk \leq \sum_{k=1}^{k=\lceil \frac{n}{2} \rceil - 1} klgk$ + lgn$\sum_{k=\lceil \frac{n}{2} \rceil}^{k=n-1} k$
- What next?

  The lg k in the first term is bounded by lg n/2

  $\sum_{k=1}^{k=n-1} klgk \leq \sum_{k=1}^{k=\lceil \frac{n}{2} \rceil - 1} klg\frac{n}{2}$ + lgn$\sum_{k=\lceil \frac{n}{2} \rceil}^{k=n-1} k$

Tightly Bounding (cont'd)

- $\sum_{k=1}^{k=n-1} klgk \leq \sum_{k=1}^{k=\lceil\frac{n}{2}\rceil-1} klg\frac{n}{2}$ + lgn$\sum_{k=\lceil\frac{n}{2}\rceil}^{k=n-1} k$
- What next?

  lg n/2 = lg n - 1

  $\sum_{k=1}^{k=n-1} klgk \leq \sum_{k=1}^{k=\lceil\frac{n}{2}\rceil-1} k(lgn-1)$ + lgn$\sum_{k=\lceil\frac{n}{2}\rceil}^{k=n-1} k$

Tightly Bounding (cont'd)

- $\sum_{k=1}^{k=n-1} klgk \leq \sum_{k=1}^{k=\lceil\frac{n}{2}\rceil-1} k(lgn-1)$ + lgn$\sum_{k=\lceil\frac{n}{2}\rceil}^{k=n-1} k$
- What next?
  - Move (lg n - 1) outside the summation

$$\sum_{k=1}^{k=n-1} klgk \leq (lgn-1)\sum_{k=1}^{k=\lceil\frac{n}{2}\rceil-1} k + \text{lgn}\sum_{k=\lceil\frac{n}{2}\rceil}^{k=n-1} k$$

Tightly Bounding (cont'd)

- $\sum_{k=1}^{k=n-1} klgk \leq (lgn - 1) \sum_{k=1}^{k=\lceil \frac{n}{2} \rceil - 1} k$+ lgn$\sum_{k=\lceil \frac{n}{2} \rceil}^{k=n-1} k$
- What next?
  - Distribute the (lg n - 1)

$$\sum_{k=1}^{k=n-1} klgk \leq lgn \sum_{k=1}^{k=\lceil \frac{n}{2} \rceil - 1} k - \sum_{k=1}^{k=\lceil \frac{n}{2} \rceil - 1} k + \text{lgn} \sum_{k=\lceil \frac{n}{2} \rceil}^{k=n-1} k$$

Tightly Bounding (cont'd)

- $\sum_{k=1}^{k=n-1} klgk \leq lgn \sum_{k=1}^{k=\left\lceil \frac{n}{2} \right\rceil -1} k - \sum_{k=1}^{k=\left\lceil \frac{n}{2} \right\rceil -1} k + \lg n \sum_{k=\left\lceil \frac{n}{2} \right\rceil}^{k=n-1} k$
- What next?
  - The summations overlap in range; combine them

$$\sum_{k=1}^{k=n-1} klgk \leq lgn \sum_{k=1}^{k=n-1} k - \sum_{k=1}^{k=\left\lceil \frac{n}{2} \right\rceil -1} k$$

Tightly Bounding (cont'd)

- $\sum_{k=1}^{k=n-1} klgk \leq lgn \sum_{k=1}^{k=n-1} k - \sum_{k=1}^{k=\left\lceil \frac{n}{2} \right\rceil - 1} k$
- What next?
  - The Gaussian series

$$\sum_{k=1}^{k=n-1} klgk \leq \lg\left(\frac{(n-1)n}{2}\right) - \sum_{k=1}^{k=\left\lceil \frac{n}{2} \right\rceil - 1} k$$

Tightly Bounding (cont'd)

- $\sum_{k=1}^{k=n-1} klgk \leq \lg(\frac{(n-1)n}{2}) - \sum_{k=1}^{k=\lceil \frac{n}{2} \rceil - 1} k$
- What next?
  - Rearrange first term, place upper bound on second

$$\sum_{k=1}^{k=n-1} klgk \leq \frac{1}{2}[n(n-1)]lgn - \sum_{k=1}^{k=n/2-1} k$$

Tightly Bounding (cont'd)

- $\sum_{k=1}^{k=n-1} klgk \leq \frac{1}{2}[n(n-1)]lgn - \sum_{k=1}^{k=n/2-1} k$
- What next?
  - The Gaussian series
  - $\sum_{k=1}^{k=n-1} klgk \leq \frac{1}{2}[n(n-1)]lgn - \frac{1}{2}\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)$

Tightly Bounding (cont'd)

- $\sum_{k=1}^{k=n-1} klgk \leq \frac{1}{2}[n(n-1)]lgn - \frac{1}{2}\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)$
- What next?
  - Multiply it all out
  - $\sum_{k=1}^{k=n-1} klgk \leq \frac{1}{2}(n^2 lgn - nlgn)$ - $\frac{1}{8}\ n^2\ +\ \frac{n}{4}$

$$= \frac{1}{2}n^2 lgn - \frac{1}{8}\ n^2 - \frac{1}{2}nlgn + \frac{n}{4}$$

$$\leq \frac{1}{2}n^2 lgn - \frac{1}{8}\ n^2 \text{ when n} \geq 2$$

Done !!!!

Tightly Bounding Summary

$$\sum_{k=1}^{k=n-1} klgk = \sum_{k=1}^{k=\lceil n/2\rceil-1} klgk + \sum_{k=\lceil n/2\rceil}^{k=n-1} klgk$$

$$\leq \sum_{k=1}^{k=\lceil n/2\rceil-1} klg(\frac{n}{2}) + \sum_{k=\lceil n/2\rceil}^{k=n-1} klgn$$

$$= \sum_{k=1}^{k=\lceil n/2\rceil-1} k(lgn-1) + \text{lgn} \sum_{k=\lceil n/2\rceil}^{k=n-1} k$$

$$= (lgn-1) \sum_{k=1}^{k=\lceil n/2\rceil-1} k + \text{lgn} \sum_{k=\lceil n/2\rceil}^{k=n-1} k$$

Tightly Bounding Summary (cont'd)

$$\sum_{k=1}^{k=n-1} klgk \leq (lgn-1)\sum_{k=1}^{k=\lceil n/2\rceil-1} k + \lg n \sum_{k=\lceil n/2\rceil}^{k=n-1} k$$

$$= lgn \sum_{k=1}^{k=\lceil n/2\rceil-1} k - \sum_{k=1}^{k=\lceil n/2\rceil-1} k + \lg n \sum_{k=\lceil n/2\rceil}^{k=n-1} k$$

$$= lgn \sum_{k=1}^{k=n-1} k - \sum_{k=1}^{k=\lceil n/2\rceil-1} k$$

$$= \lg n \left(\frac{(n-1)n}{2}\right) - \sum_{k=1}^{k=\lceil n/2\rceil-1} k$$

$$\leq \frac{1}{2}[n(n-1)]lgn - \sum_{k=1}^{k=n/2-1} k$$

Tightly Bounding Summary(cont'd)

$$\sum_{k=1}^{k=n-1} klgk \leq \frac{1}{2}[n(n-1)]lgn - \sum_{k=1}^{k=n/2-1} k$$

$$= \frac{1}{2}[n(n-1)]lgn - \frac{1}{2}\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)$$

$$= \frac{1}{2}(n^2lgn - nlgn) - \frac{1}{8}n^2 + \frac{n}{4}$$

$$= \frac{1}{2}n^2lgn - \frac{1}{8}n^2 - \frac{1}{2}nlgn + \frac{n}{4}$$

$$\leq \frac{1}{2}n^2lgn - \frac{1}{8}n^2 \text{ when } n \geq 2$$

Done !!!!